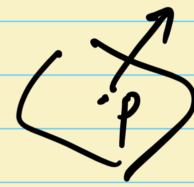
Geometric objects in $\mathbb{R}^n$

- lines in $\mathbb{R}^n$: $\vec{x} = \vec{a} + t\vec{v} \quad t \in \mathbb{R}.$
- planes in $\mathbb{R}^3$: $\left[\begin{array}{l} \vec{x} = \vec{a} + s\vec{u} + t\vec{v} \quad s, t \in \mathbb{R} \\ ax + by + cz = d \end{array}\right.$



- line in $\mathbb{R}^3$ — parametric form
  - 2 equations (i.e. intersection of 2 planes)

Intersection of 3 planes (in terms of intersections)

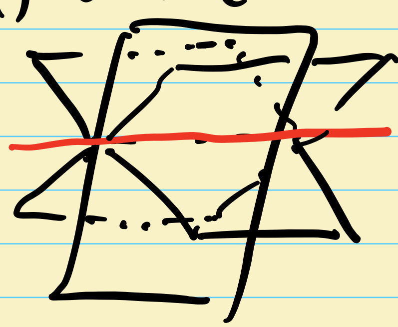
case 1) unique solution

e.g. xy-plane, yz-plane, zx-plane in $\mathbb{R}^3$

$\Rightarrow$ intersection $= (0, 0, 0)$

case 2) infinitely many solutions

i) a line



e.g. $x = 0$

$z = 0$

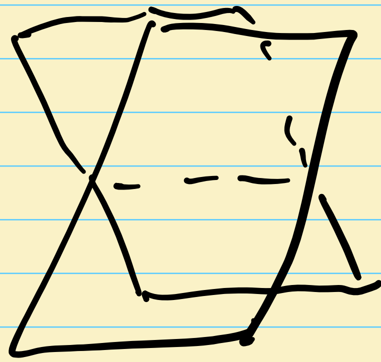
$x + z = 0$

$\Rightarrow$ intersection

y-axis

ii) a line

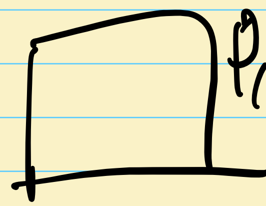
P_1, P, P_3 $\quad$ $P_2 = P_3$



$P_2 = P_3$

iii) a plane

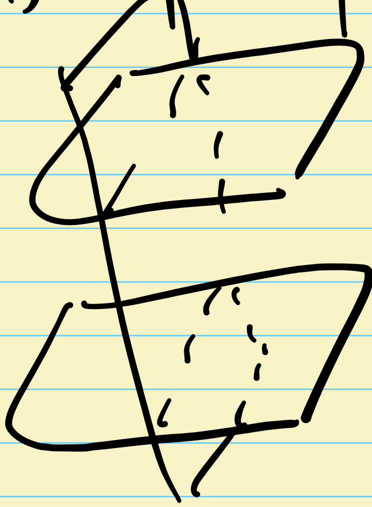
$P_1 = P_2 = P_3$

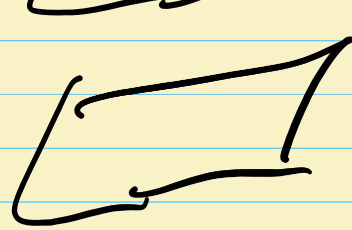


$P_1 = P_2 = P_3$

case 3) no solution (i.e. no common intersection.)

i) a pair parallel $\quad$ ii) all parallel $\quad$ iii) no parallel each other





General linear objects in $\mathbb{R}^n$

In $\mathbb{R}^3$

1 equation $\leftrightarrow$ a plane

2 equations $\leftrightarrow$ a line

In $\mathbb{R}^4$ or higher

1 equn $\leftrightarrow$ hyperplane

2 eqns $\leftrightarrow$ a plane

3 equns $\leftrightarrow$ line

In $\mathbb{R}^n$, $\vec{a}\cdot\vec{x} = c$ $(\vec{a} \neq 0)$ i.e.

$a_1x_1 + \cdots + a_nx_n = c$ describes a hyperplane in $\mathbb{R}^n$. (normal vector is $\vec{a}$).

In $\mathbb{R}^1$ a point = a hyperplane

- In $\mathbb{R}^2$ a line = a hyperplane

In $\mathbb{R}^3$ a plane = a hyperplane

- $c = 0$; $\vec{a}\cdot\vec{x} = 0$ is a hyperplane consisting of vectors orthogonal to $\vec{a}$.

- dimension of a hyperplane in $\mathbb{R}^n$ $= n-1$.

- To describe a k-dimensional "plane" (called k-plane) in $\mathbb{R}^n$.

i) parametric form

(Recall. in $\mathbb{R}^3$ $\vec{a} + t\vec{v} + s\vec{w}$)

$$\vec{x} = \vec{a} + \sum_{i=1}^{k} t_i \vec{v}_i$$

- $\vec{a}$ a point, - $\vec{v}_1, \cdots, \vec{v}_k$ are linearly independent vects

· $t_1, \dots, t_k$ are parameters

iii) equations $\left\{ \begin{array}{l} \text{In } \mathbb{R}^3 \\ \text{line} \to \dim 1 \to (3-1) \text{ equs} \\ \text{plane} \to \dim 2 \to (3-2) \quad " \end{array} \right)$

$(a_{11}, \dots, a_{1n})$
$\vdots$
$(a_{n-k\,1}, \dots, a_{n-k\,n})$
are linearly independent

$\{$ $n-k$ non-redundant equations

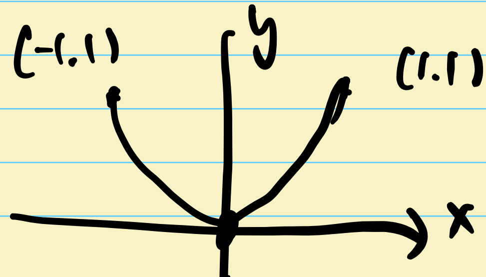
$$\sum_{j=1}^{n} a_{ij} x_j = c_i \quad (i = 1, \dots, n-k)$$

i.e. intersection of $n-k$ hyperplanes.

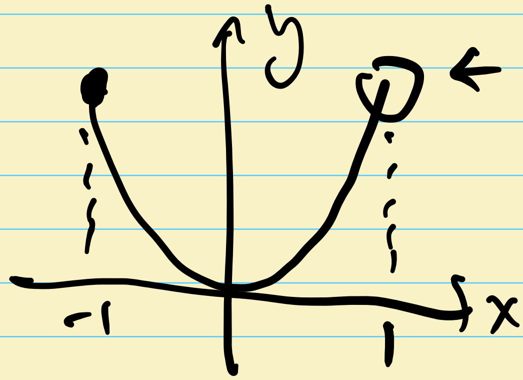
Curves in $\mathbb{R}^n$

Def Let $I \subset \mathbb{R}$ be an interval. A curve in $\mathbb{R}^n$ is a continuous function $\vec{x} : I \to \mathbb{R}^n$.
i.e. $\vec{x}(t) = (x_1(t), \dots, x_n(t))$ such that every $x_i(t)$ is continuous.

eg $\vec{x} : [-1, 1] \longrightarrow \mathbb{R}^2$ defined by
$\vec{x}(t) = (t, t^2) \rightsquigarrow y = x^2$ a parabola

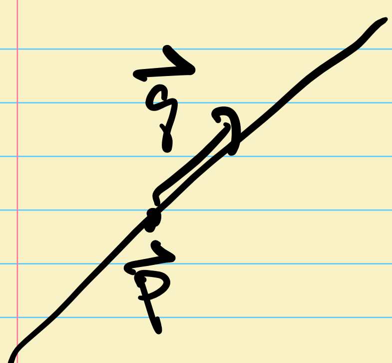


$\vec{x} : [-1, 1) \to \mathbb{R}^2 \qquad \vec{x}(t) = (t, t^2)$



eg $\vec{x} : \mathbb{R} \to \mathbb{R}^3$.

$\vec{x}(t) = \vec{p} + t\vec{q}$ where $\vec{p}, \vec{q} \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

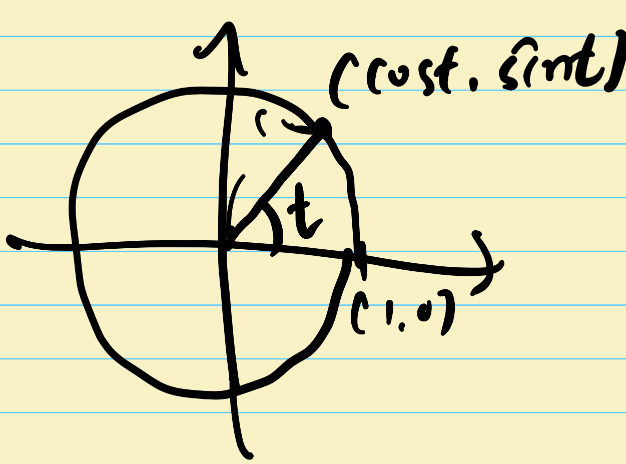


a straight line.

eg $\vec{x} : [0, 2\pi] \to \mathbb{R}^2$ define by

$\vec{x}(t) = (\cos t, \sin t)$

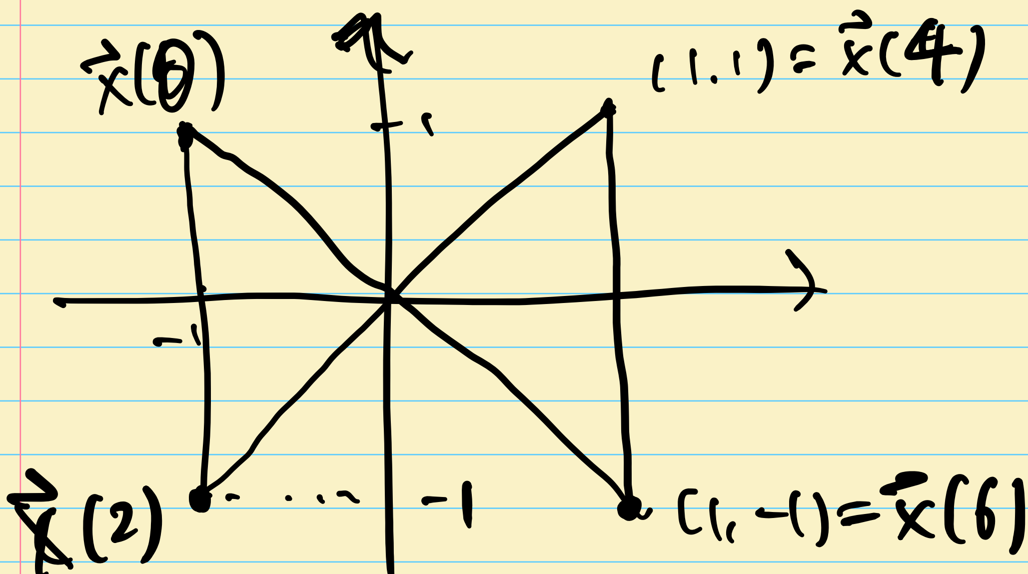
$\leadsto x^2 + y^2 = 1$.



a circle of radius 1 centered at the origin.

eg $\vec{x} : [0,8] \longrightarrow \mathbb{R}^2$ defined by

$$\vec{x}(t) = \begin{cases} (-1,\ 1-t) & 0 \le t \le 2 \\ (t-3,\ t-3) & 2 \le t \le 4 \\ (1,\ 5-t) & 4 \le t \le 6 \\ (7-t,\ t-7) & 6 \le t \le 8 \end{cases}$$



Def A curve $\vec{x} : [a,b] \longrightarrow \mathbb{R}^n$ is said to be

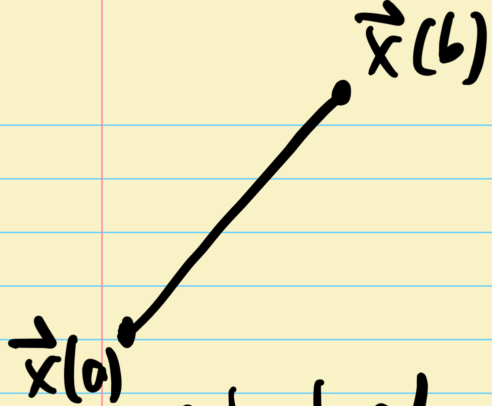
i) <u>closed</u> if $\vec{x}(a) = \vec{x}(b)$

ii) <u>simple</u> if $\vec{x}(t_1) \neq \vec{x}(t_2)$ for any $a \le t_1 < t_2 \le b$. except possibly at $t_1 = a$, $t_2 = b$.

eg

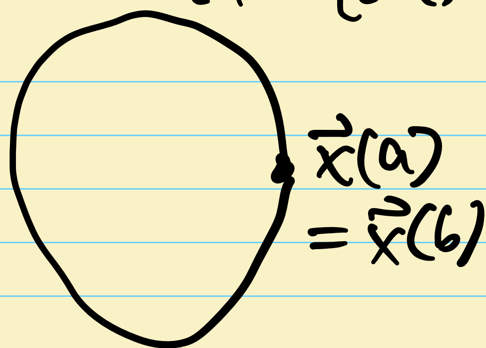
$[0, 2\pi] \to \mathbb{R}^2$

$t \longmapsto (\cos t, \sin t)$



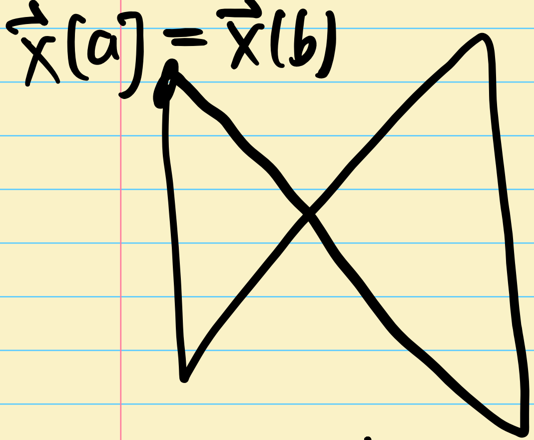
not closed

simple



closed

simple

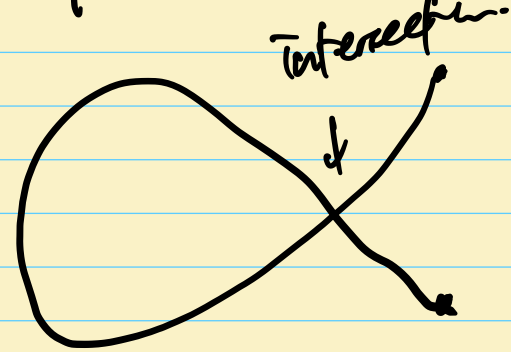


closed

not simple

$t_1 = 3, \quad t_2 = 7$

$\vec{x}(t_1) = (0,0) = \vec{x}(t_2)$



not closed

not simple

Def Let $\vec{x}(t) = (x_1(t), \ldots, x_n(t))$ be a curve.

The derivative of $\vec{x}$ at t is

$$\vec{x}\,'(t) = (x_1'(t), \ldots, x_n'(t))$$

notation $\lim_{t\to a} \vec{x}(t) = \left(\lim_{t\to a} x_1(t), \dots, \lim_{t\to a} x_n(t)\right)$

$$\vec{x}'(t) = \lim_{h\to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}$$

$$= \left(\lim_{h\to 0} \frac{x_1(t+h)-x_1(t)}{h}, \dots, \lim_{h\to 0} \frac{x_n(t+h)-x_n(t)}{h}\right)$$

$$= (x_1'(t), \dots, x_n'(t))$$

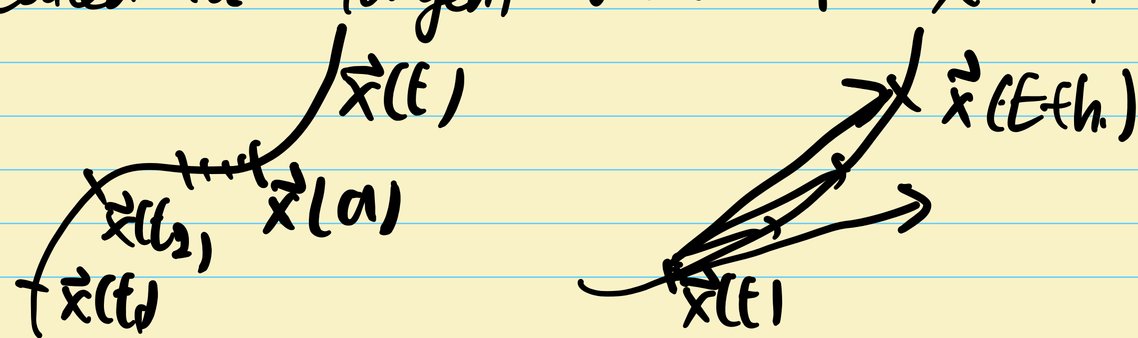
$\times$: $\vec{x}'(t)$ may not exist

eg $\vec{x}(t) : (-1,1) \longrightarrow \mathbb{R}^2$

$\vec{x}(t) = (|t|, t)$

For any a in the domain of $\vec{x}$ $(a \in I)$ if $\vec{x}'(a)$ exists, then $\vec{x}'(a)$ is called the tangent vector of $\vec{x}$ at $t=a$.

Picture



If we interpret t in $\vec{x}(t)$ as time,

$\vec{x}(t)$: a position of a particle moving in $\mathbb{R}^n$ at time t.

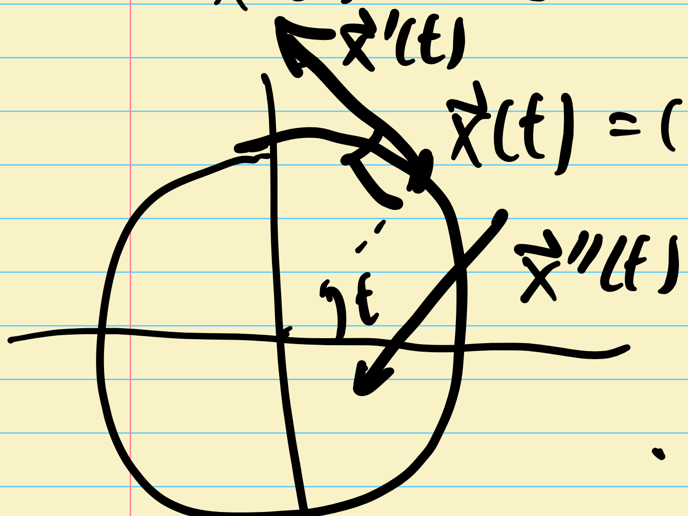
$\vec{x}'(t)$: velocity of the particle at t. $(= \vec{v}(t))$

$\vec{x}''(t)$: acceleration $(= \vec{a}(t))$

Example $\vec{x}(t) = (\cos t, \sin t)$ $t \in [0, 2\pi]$

$\vec{x}'(t) = (-\sin t, \cos t) = \vec{v}(t)$

$\vec{x}''(t) = (-\cos t, -\sin t) = \vec{a}(t)$



- $\vec{v}(t)$ is orthogonal to $\vec{x}(t)$
- $\vec{a}(t) = -\vec{x}(t)$
- $\|\vec{v}(t)\| = 1$

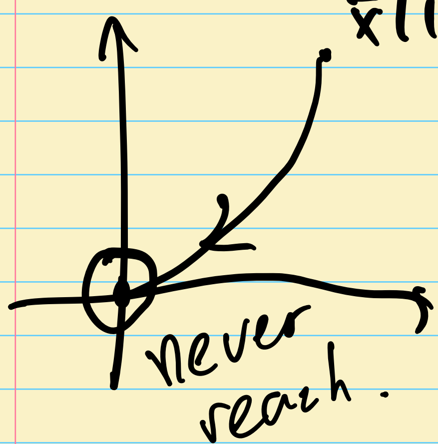
Example $\vec{x} : [1, \infty) \longrightarrow \mathbb{R}^2$ defined by

$$\vec{x}(t) = \left(\frac{1}{t}, \frac{1}{t^2}\right)$$

x y

$\frac{1}{t} \neq 0$

$y = x^2$



$$\lim_{t\to\infty} \vec{x}(t)$$

$$= \left(\lim_{t\to\infty} \frac{1}{t}, \lim_{t\to\infty} \frac{1}{t^2}\right)$$

$$= (0, 0)$$

Prop Let $\vec{x}(t)$, $\vec{y}(t)$ be curves in $\mathbb{R}^n$, $c \in \mathbb{R}$ a constant, $f(t) : \mathbb{R} \to \mathbb{R}$ a real-valued function.

① $(\vec{x}(t) \pm \vec{y}(t))' = \vec{x}'(t) \pm \vec{y}'(t)$

② $(c\vec{x}(t))' = c\vec{x}'(t)$

③ $\left(f(t)\vec{x}(t)\right)' = f'(t)\vec{x}(t) + f(t)\vec{x}'(t)$

④ $(\vec{x}(t) \cdot \vec{y}(t))' = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t)$

⑤ (In $\mathbb{R}^3$) $\left(\vec{x}(t) \times \vec{y}(t)\right)' = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t)$.

(proof) ③ ~ ⑤ ~~are~~ all follows from

$$(f(t)\,g(t))' = f'(t)\,g(t) + f(t)\,g'(t)$$

in one-variable calculus

e.g. ③

$$(f(t)\vec{x}(t))' = (f(t)x_1(t), \cdots, f(t)x_n(t))'$$

$$= ((\;''\;)', \cdots, (\;''\;)')$$

$$= (f'(t)x_1(t) + f(t)\cdot x_1'(t),$$

$$\vdots$$

$$f'(t)x_n(t) + f(t)\,x_n'(t))$$

$$= f'(t)\cdot(x_1(t), \cdots, x_n(t)) + f(t)(x_1'(t), \cdots x_n'(t))$$

$$= f'(t)\vec{x}(t) + f(t)\vec{x}'(t) \qquad \square$$

Arclength

Let $\vec{x}(t): [a,b] \to \mathbb{R}^n$ be a curve.

Suppose $\vec{x}\,'(t)$ exists and is continuous on (a,b).

Def The arclength of $\vec{x}$ on $[a,b]$ is

$$S = \int_a^b \|\vec{x}\,'(t)\|\, dt$$

$\vec{x}(t)$ = position at t

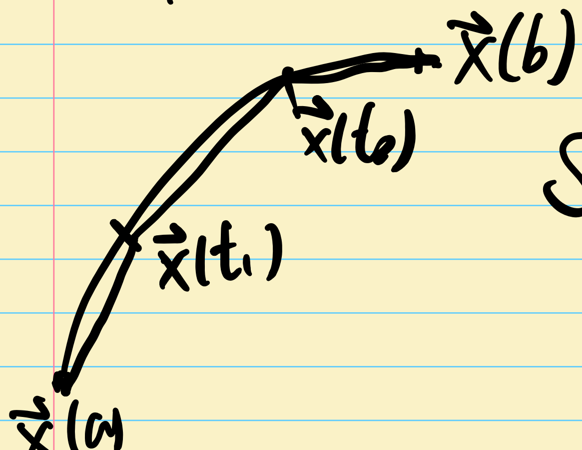
$\vec{x}\,'(t)$ = velocity

$\|\vec{x}\,'(t)\|$ = speed

$\int_a^b \|\vec{x}\,'(t)\|\, dt$ = distance travelled.

More rigorously,

approximate a curve by line segments:



Take $a = t_0 < t_1 < \cdots < t_n = b$

$S \approx$ sum of length of line segments

$$= \sum_{i=1}^{n} \|\vec{x}(t_i) - \vec{x}(t_{i-1})\|$$

$$\vec{x}'(t_i) = \lim_{t \to t_i} \frac{\vec{x}(t) - \vec{x}(t_i)}{t - t_i}$$

$$\approx \sum_{i=1}^{n} \|\vec{x}'(t_i)\| (t_i - t_{i-1})$$

Take limits

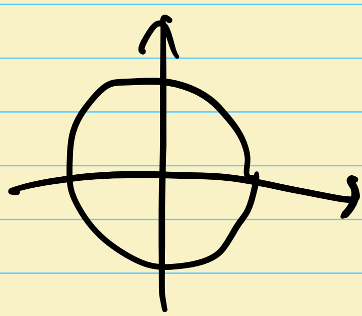
$$\Rightarrow \quad S = \int_a^b \|\vec{x}'(t)\| dt.$$

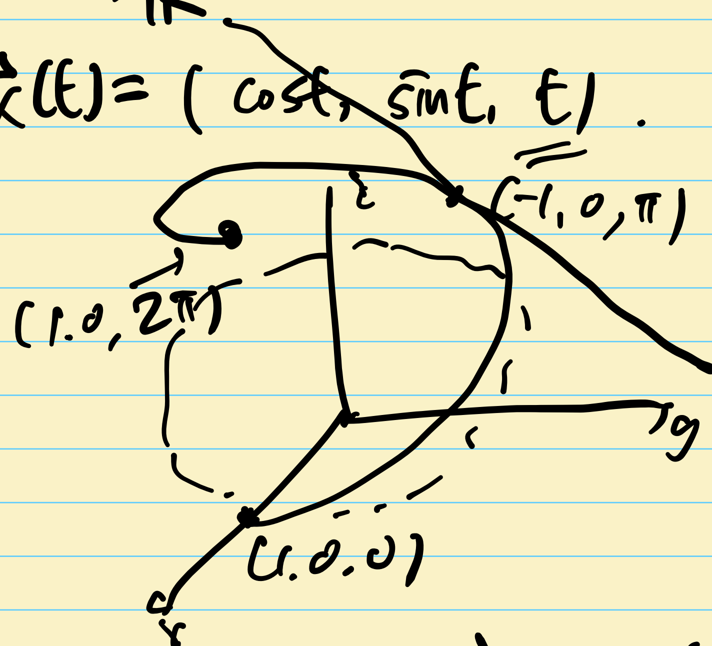
<u>Example</u> (Helix)

$\vec{x} : [0, 2\pi] \to \mathbb{R}^3$

defined by $\vec{x}(t) = (\cos t, \sin t, t)$.

(x,y)-plane





(a) Find the <u>tangent line</u> of $\vec{x}$ at $t = \pi$.

(b) Find the arclength of $\vec{x}$.

(sol) (a) pass thru $\vec{x}(\pi) = (-1, 0, \pi)$

direction? $\vec{x}'(t) = (-\sin t, \cos t, 1)$

at $t = \pi$.

$\vec{x}'(\pi) = (0, -1, 1)$

$\therefore$ In parametric form. $(-1, 0, \pi) + t(0, -1, 1)$

(b) $\|\vec{x}'(t)\| = \|(-\sin t, \cos t, 1)\|$

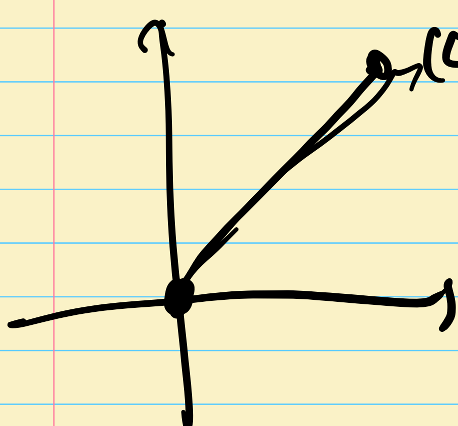
$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

$$\therefore S = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi. \quad \square$$

Example. $\vec{x}(t) = (t,t) \quad 0 \le t \le 4$

$\vec{y}(t) = (t^2, t^2) \quad 0 \le t \le 2.$



arclength using $\vec{x}$;

$\vec{x}'(t) = (1,1)$

arclength of $\vec{x}(t)$

$$= \int_0^4 \|\vec{x}'(t)\|\, dt$$

$$= \int_0^4 \sqrt{2}\, dt = 4\sqrt{2}$$

arclength using $\vec{y}$;

$\vec{y}'(t) = (2t, 2t)$

arclength of $\vec{y}(t) = \int_0^2 \|\vec{y}'(t)\| dt$

$$= \int_0^2 \sqrt{(2t)^2 + (2t)^2}\, dt$$

$$= \int_0^2 2\sqrt{2}\, t\, dt$$

$$= \left[\sqrt{2}\, t^2\right]_0^2$$

$$= 4\sqrt{2}$$

Thm Arclength is independent of parametrization.

(proof) What does it mean by two parametrizations are equal?

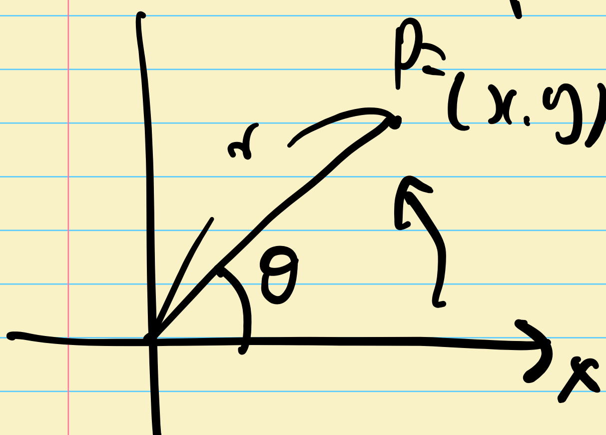
After we learn change of variables for integration ... :D

Polar coordinates

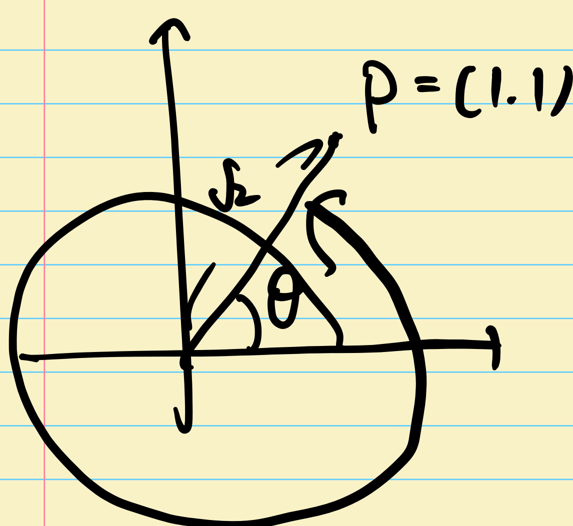
A point $P=(x,y)$ in $\mathbb{R}^2$ can be represented by

$r = \sqrt{x^2+y^2}$ = distance from origin

θ = angle from the positive x-axis to $\overrightarrow{OP}$ in counter-clockwise direction



eg $P=(1,1)$



$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \frac{\pi}{4}$ or $\frac{\pi}{4}+2\pi$

or $\frac{\pi}{4}+4\pi$

or $\frac{\pi}{4}-2\pi$.

$= \frac{\pi}{4} + (2\pi)k$

$k \in \mathbb{Z}$

Rmk

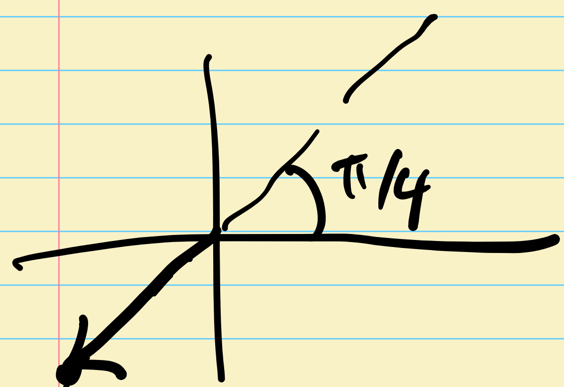
- $P \neq 0$; r is unique but
  θ is not uniquely determined
  (up to $2k\pi$, $k \in \mathbb{Z}$)

- $P = (0,0)$; $r = 0$
  θ is not uniquely determined.

- $r \in [0, \infty)$ or $r \in \mathbb{R}$

  i.e.

  $r = -\sqrt{2}$, $\theta = \frac{\pi}{4}$; $(-1, -1)$



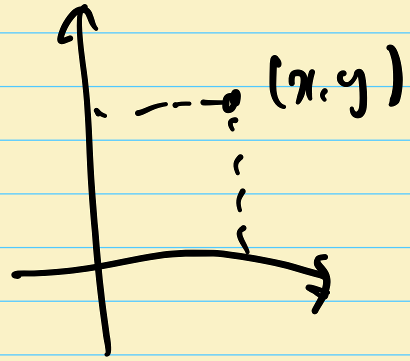
We will use
$r \in [0, \infty)$

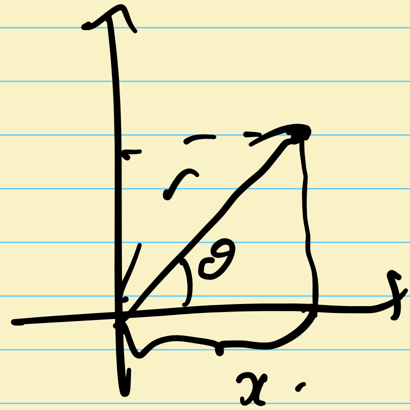
- $\theta \in [0, 2\pi)$ or $\theta \in \mathbb{R}$

  We will use $\theta \in \mathbb{R}$.

Change of coordinates

Point





From (r, θ), $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

From (x, y), $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$ if $x, y > 0$

i.e. the point lies in the 1st quadrant

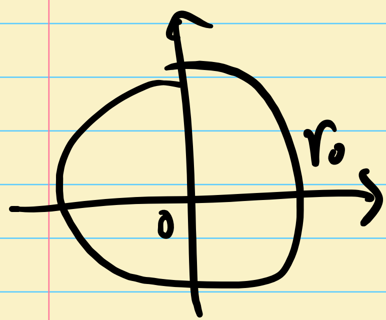
similar formulas for other quadrants,

eg $x<0$, $y>0$

$\theta = \pi + \tan^{-1}\left(\frac{y}{x}\right)$

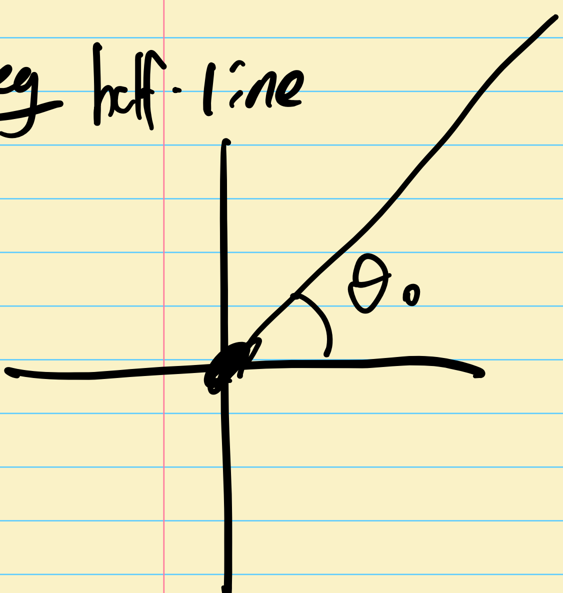
curve (line, circle etc.)

eg circle



	xy-coord	polar coord
equations	$x^2+y^2=r_0^2$	$r=r_0$
parametrization	(x,y) $=(r_0\cos t, r_0\sin t)$ $0\leq t\leq 2\pi$	(r,θ) $=(r,t)$ $0\leq t\leq 2\pi$ $0\leq t<\infty$

eg half-line



	xy-coord	polar coord
equation	?	$\theta=\theta_0$